THE γ-FILTRATION AND THE ROST INVARIANT

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Abstract. Let X be the variety of Borel subgroups of a simple and strongly inner linear algebraic group G over a field k. We prove that the torsion part of the second quotient of Grothendieck's γ-filtration on X is a cyclic group of order the Dynkin index of G.

As a byproduct of the proof we obtain an explicit cycle θ that generates this cyclic group; we provide an upper bound for the torsion of the Chow group of codimension-3 cycles on X; we relate the cycle θ with the Rost invariant and the torsion of the respective generalized Rost motives; we use θ to obtain a uniform lower bound for the essential dimension of (almost) all simple linear algebraic groups.

Grothendieck's celebrated γ-*filtration* of the ring $K_0(X)$ gives a way to estimate the Chow group $\mathrm{CH}^*(X)$ of algebraic cycles on X modulo rational equivalence when X is a smooth projective variety over a field k. Namely, by the Riemann-Roch theorem without denominators [8, §15.3] the i-th *Chern class* provides a well-defined group homomorphism

$$c_i\colon \gamma^{i/i+1}(X) \to \mathrm{CH}^i(X), \quad i \geq 0$$

from the i-th quotient of the γ-filtration to the Chow group of codimension-i cycles on X. Observe that for $i = 0$ it is the identity map and for $i = 1$ it is an isomorphism identifying $\mathrm{CH}^1(X)$ with the Picard group of X.

In the present paper we study these homomorphisms in the cases $i = 2, 3$ and X is a generically split projective homogeneous variety under a semisimple linear algebraic group G. Our core determines and bounds respectively the torsion subgroup of $\gamma^{2/3}(\mathfrak{B})$ and $\gamma^{3/4}(\mathfrak{B})$ for the variety of Borel subgroups $\mathfrak{B}$ of strongly inner G (Theorem 3.1). For instance, we show that the torsion subgroup $\mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B})$ is cyclic of order the Dynkin index of G and exhibit a generator θ for it (Definition 3.3).

This fact together with the Riemann-Roch theorem imply (see §4) that the surjection

$$\langle\theta\rangle = \mathrm{Tors}\,\gamma^{2/3}(\mathfrak{B}) \xrightarrow{c_2} \mathrm{Tors}\,\mathrm{CH}^2(\mathfrak{B})$$

can be viewed as a substitute of the key map $Q(V) \to H^3(k, \mathbb{Q}/\mathbb{Z}(2))$ in the definition of the Rost invariant [10, pp. 126-127]. Indeed, a theorem of Peyre-Merkurjev [27] shows that $\mathrm{Tors}\,\mathrm{CH}^2(\mathfrak{B})$ can be identified with the kernel of the restriction $H^3(k, \mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(\mathfrak{B}), \mathbb{Q}/\mathbb{Z}(2))$. Furthermore, the order of $c_2(\theta)$ in $\mathrm{Tors}\,\mathrm{CH}^2(\mathfrak{B})$ equals to the order of the Rost invariant of G (see Prop. 3.2).

Our result gives bounds for the torsion in CH^3 for generically split X (see §5) and provides explicit generators of torsion subgroups of CH^2 of certain generalized Rost-Voevodsky motives. Note that typically, one does not even know a priori if the torsion subgroup of $\mathrm{CH}^i(X)$, $i \geq 3$, is finitely generated. However, determining the torsion subgroup determines $\mathrm{CH}^i(X)$ as an abelian group, since the dimension of its free part $\mathrm{CH}^i(X) \otimes \mathbb{Q}$ can be easily computed.

In section 6, we study the behaviour of the image of c_2 under field extensions. In particular, we show that this image is non-trivial if G has Tits algebras of index 2 (Prop. 6.5). Using this fact we prove (Prop. 7.1) that the essential dimension $\mathrm{ed}(G)$ of any absolutely almost simple linear algebraic group not of type A nor isomorphic to Sp_{2n} is greater or equal than 3.

1. Preliminaries

We now provide several facts and observations concerning Chow groups, characteristic maps, invariants, Dynkin indices and filtrations on K_0 for varieties of Borel subgroups of split simple linear algebraic groups.

§1A. **Two filtrations on K_0.** All facts provided here can be found in [17, §2], [8, §15] and [9, Ch. 3,5]. Let X be a smooth projective variety over a field k. Consider the γ-filtration on $K_0(X)$. It is given by the subgroups

$$\gamma^i(X) = \langle c_{n_1}^{K_0}(b_1)\cdots c_{n_m}^{K_0}(b_m) \mid n_1+\cdots+n_m \geq i \text{ and } b_1,\ldots,b_m \in K_0(X)\rangle,$$

where $c_n^{K_0}$ denote the n-th Chern class with values in K_0. For example, for the class of a line bundle we have $c_1^{K_0}([\mathcal{L}]) = 1-[\mathcal{L}^*]$. Let $\gamma^{i/i+1}(X) = \gamma^i(X)/\gamma^{i+1}(X)$ denote the respective quotient. Consider the topological filtration on $K_0(X)$ given by the subgroups

$$\tau^i(X) = \langle [\mathcal{O}_V] \mid V \hookrightarrow X \text{ and } \operatorname{codim} V \geq i\rangle,$$

where $[\mathcal{O}_V]$ is the class of the structure sheaf of a closed subvariety V. Let $\tau^{i/i+1}(X) = \tau^i(X)/\tau^{i+1}(X)$ denote the corresponding quotient.

There is an obvious surjection $\mathrm{pr}\colon \mathrm{CH}^i(X) \twoheadrightarrow \tau^{i/i+1}(X)$ from the Chow group of codimension i cycles given by $V \mapsto [\mathcal{O}_V]$. By the Riemann-Roch Theorem without denominators the i-th Chern class induces the map in the opposite direction

$$c_i\colon \tau^{i/i+1}(X) \to \mathrm{CH}^i(X)$$

and the composite $c_i \circ \mathrm{pr}$ is the multiplication by $(-1)^{i-1}(i-1)!$ which is an isomorphism for $i \leq 2$ [8, Ex.15.3.6]. For example, by the very definition we have

$$c_i(\prod_{j=1}^{i} c_1^{K_0}([\mathcal{L}_j])) = (-1)^{i-1}(i-1)!\prod_{j=1}^{i} c_1^{\mathrm{CH}}(\mathcal{L}_j),$$

where $\mathcal{L}_j$ is a line bundle. Observe also that c_i becomes an isomorphism after tensoring with $\mathbb{Q}$.

There is an embedding $\gamma^i(X) \subset \tau^i(X)$ for all i. Moreover, $\gamma^i(X) = \tau^i(X)$ for $i \leq 2$. Observe that $\gamma^{1/2}(X) = \tau^{1/2}(X) = \mathrm{CH}^1(X)$ is the Picard group and by [17, Cor. 2.15] there is an exact sequence

$$0 \to \tau^3(X)/\gamma^3(X) \to \operatorname{Tors}\gamma^{2/3}(X) \xrightarrow{c_2} \operatorname{Tors}\mathrm{CH}^2(X) \to 0, \tag{1.1}$$

where we have written c_2 for the composition $\gamma^{2/3}(X) \to \tau^{2/3}(X) \xrightarrow{c_2} \mathrm{CH}^2(X)$.

§1B. **Characteristic maps and invariants.** Let G_s be a split simply connected simple linear algebraic group of rank n over a field k. We fix a split maximal torus T and a Borel subgroup B such that $T \subset B \subset G_s$. Let $\mathfrak{B}_s$ denote the variety of Borel subgroups of G_s and let T^* denote the group of characters of T. We fix a basis of T^* given by the fundamental weights $\omega_1, \ldots, \omega_n$.

Let $\mathbb{S}(T^*)$ be the symmetric algebra of T^*. Its elements are polynomials in the fundamental weights ω_i with coefficients in $\mathbb{Z}$. Let $\mathbb{Z}[T^*]$ be the integral group ring of T^*. Its elements are integral linear combinations $\sum_i a_i e^{\lambda_i}$, $\lambda_i \in T^*$. Consider the characteristic maps for CH and K_0 (see [5, §8, 9] and [6, §1.5, 1.6])

$$\mathfrak{c}\colon\ \mathbb{S}(T^*) \to \mathrm{CH}(\mathfrak{B}_s) \text{ and } \mathfrak{c}'\colon \mathbb{Z}[T^*] \to K_0(\mathfrak{B}_s)$$

given by

$$\mathfrak{c}\colon \omega_i \mapsto c_1^{\mathrm{CH}}(\mathcal{L}(\omega_i)) \text{ and } \mathfrak{c}'\colon e^{\lambda} \mapsto [\mathcal{L}(\lambda)]$$

where $\mathcal{L}(\lambda)$ is the line bundle over $\mathfrak{B}_s$ associated to the character λ.

There are obvious augmentation maps $\mathbb{S}(T^*) \to \mathbb{Z}$ and $\mathrm{aug}\colon \mathbb{Z}[T^*] \to \mathbb{Z}$ given by $\omega_i \mapsto 0$ and $e^{\lambda} \mapsto 1$ respectively. The Weyl group acts naturally on T^*, hence also on $\mathbb{S}(T^*)$ and $\mathbb{Z}[T^*]$. Consider the subrings of invariants $\mathbb{S}(T^*)^W$ and $\mathbb{Z}[T^*]^W$. Let I (resp. I') be the ideal generated by the elements of $\mathbb{S}(T^*)^W$ (resp. $\mathbb{Z}[T^*]^W$) from the kernel of the augmentation map. Then we have

$$\ker \mathfrak{c} = I \text{ and } \ker \mathfrak{c}' = I'.$$

Therefore we have embeddings

$$\mathfrak{c}\colon\ \mathbb{S}(T^*)/I \hookrightarrow \mathrm{CH}(\mathfrak{B}_s) \text{ and } \mathfrak{c}'\colon \mathbb{Z}[T^*]/I' \xrightarrow{\simeq} K_0(\mathfrak{B}_s),$$

where the second map is surjective since G_s is simply connected (see [28]).

By [5, §2 and Cor.2] the kernel I of $\mathfrak{c}$ consists of elements g such that

$$m \cdot g = \sum_i g_i \cdot f_i, \tag{1.2}$$

for $m \in \mathbb{Z}$, f_i the basic polynomial invariants, and $g_i \in \mathbb{S}(T^*)$.

There is a W-invariant quadratic form q on $T^* \otimes \mathbb{Q}$ that is uniquely determined up to a scalar multiple [2, §VI.1.1–2]. We normalize q so that it takes the value 1 on every short coroot; as q is indivisible, it can be taken as the generator of I of degree 2. To say it differently, each element of I of degree 2 is a multiple of q by an integer.

The form q should be familiar. Its polar bilinear form b_q amounts to the restriction of the "reduced Killing form" to the Cartan subalgebra of the Lie algebra of G_s as described in [15, §5]. In the case where the roots are all one length, an explicit formula for b_q is well known: its Gram matrix is the Cartan matrix of the root system.

§1C. **Degree 3 elements of I.** If G_s is not of type A_n $(n \geq 2)$, then there is no basic invariant of degree 3 [16, p. 59]. Then by (1.2) and the indivisibility of q, every $g \in I$ of degree 3 can be written as $g = (\sum a_i\omega_i)q$ for some $a_i \in \mathbb{Z}$.

If G_s is of type A_n for some $n \geq 2$, then $\mathbb{S}(T^*)^W$ has a basic generator f_3 of degree 3. We view the weight lattice as in [2], meaning that it is contained in a lattice with basis $\varepsilon_1, \ldots, \varepsilon_{n+1}$ so that the embedding is defined by $\varepsilon_1 = \omega_1$, $\varepsilon_{n+1} = -\omega_n$,

and $\varepsilon_i = \omega_i - \omega_{i-1}$ for $2 \le i \le n$, and W is the permutation group of the ε's. Then q and

$$f_3 := (\varepsilon_1^3 + \cdots + \varepsilon_{n+1}^3)/3 = \sum_{i=2}^{n} \omega_{i-1}^2\omega_i - \omega_{i-1}\omega_i^2$$

are members of a set of basic invariants [16, §3.12].

For $g \in I$ of degree 3, we have $mg = g_2 q + g_3 f_3$ for some $g_3, m \in \mathbb{Z}$ and $g_2 = \sum a_i\omega_i$ with $a_i \in \mathbb{Z}$. On the right side, the monomial ω_i^3 occurs only in $g_2 q$ and has coefficient a_i, hence m divides a_i for all i, hence m divides g_2 and also g_3. In summary, $g = (g_2/m)q + (g_3/m)f_3$ for some $g_2/m, g_3/m \in \mathbb{S}(T^*)$.

§1D. **The γ-filtration on the variety of Borel subgroups.** Consider the γ-filtration on the variety $\mathfrak{B}_s$ of Borel subgroups of G_s. Let γ^m denote the subgroup of $\mathbb{Z}[T^*]$ generated by products of at least m elements of the form $(1 - e^{-\omega_i})$, where ω_i is a fundamental weight. Then the isomorphism $\mathfrak{c}'$ induces an isomorphism

$$\gamma^{m/m+1}(\mathfrak{B}_s) \simeq \gamma^m/(\gamma^{m+1} + I') \text{ for each } i.$$

For example $\gamma^{1/2}(X) \simeq \gamma^1/(\gamma^2 + I')$ is a free abelian group with a basis given by the classes of the elements

$$(1 - e^{-\omega_i}) \in \gamma^1, \quad i = 1, \ldots, n.$$

Indeed, $c_1^{K_0}([\mathcal{L}(\omega_i)]) = 1 - [\mathcal{L}(-\omega_i)]$, the map $c_1\colon \gamma^{1/2}(\mathfrak{B}_s) \to \mathrm{CH}^1(\mathfrak{B}_s)$ is an isomorphism and the elements $c_1(\mathcal{L}(\omega_i))$ for $i = 1, \ldots, n$ form a basis of the Picard group $\mathrm{CH}^1(\mathfrak{B}_s)$.

Since $K_0(\mathfrak{B}_s)$ is generated by classes of line bundles (see [28]), so is $\gamma^i(\mathfrak{B}_s)$. Therefore, we have

$$\gamma^i(\mathfrak{B}_s) = \langle c_1^{K_0}([\mathcal{L}_1]) \cdots c_1^{K_0}([\mathcal{L}_m]) \mid m \ge i \text{ and } \mathcal{L}_j \text{ is a line bundle over } \mathfrak{B}_s\rangle.$$

Let $\lambda = \sum_i a_i\omega_i$ be a presentation of a character λ in terms of the fundamental weights. Then $\mathcal{L}(\lambda) = \otimes_i \mathcal{L}(\omega_i)^{\otimes a_i}$. Since for any two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ we have

$$c_1^{K_0}([\mathcal{L}_1 \otimes \mathcal{L}_2]) = c_1^{K_0}([\mathcal{L}_1]) + c_1^{K_0}([\mathcal{L}_2]) - c_1^{K_0}([\mathcal{L}_1])c_1^{K_0}([\mathcal{L}_2])$$

applying this formula recursively we can express any element of $\gamma^{i/i+1}(\mathfrak{B}_s)$ as a linear combination of the products of the first Chern classes of the bundles $\mathcal{L}(\omega_i)$, $i = 1 \ldots n$. For instance, any element of $\gamma^{2/3}(\mathfrak{B}_s)$ can be written as a class of

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(1 - e^{-\omega_i})(1 - e^{-\omega_j}) \in \gamma^2 \mod \gamma^3 + I', \text{ where } a_{ij} \in \mathbb{Z}.$$

§1E. **The Dynkin index.** Let N denote the map $\mathbb{Z}[T^*]^W \to \mathbb{Z}$ defined by fixing a long root α and setting

$$N\left(\sum\nolimits_i a_i e^{\lambda_i}\right) := \frac{1}{2}\sum\nolimits_i a_i \langle \lambda_i, \alpha^\vee\rangle^2.$$

This does not depend on the choice of α and takes values in $\mathbb{Z}$ (and not merely in $\frac{1}{2}\mathbb{Z}$), cf. Lemma 2.5 below. The number $N(\chi)$ is called the *Dynkin index of* χ. Note that for $m \in \mathbb{Z}$, we have $N(m) = N(me^0) = 0$, so $N(\chi)$ only depends on the image of χ in the kernel of the augmentation map.

In case G_s has two root lengths, it is natural to wonder what one would find if one used a short root, say, δ in the definition of N instead of the long root α. We claim that

$$\frac{1}{2}\sum a_i\langle\lambda_i,\delta^\vee\rangle^2 = q(\delta^\vee)\left[\frac{1}{2}\sum a_i\langle\lambda_i,\alpha^\vee\rangle^2\right], \tag{1.3}$$

where q is the form introduced in §1B. In other words, one obtains something that differs by a factor of $q(\delta^\vee)$. (We will use this observation later.) To prove it, define quadratic forms n_α and n_δ on T^* via $n_\alpha(\lambda) = \sum_{w\in W}\langle w\lambda,\alpha^\vee\rangle^2$ and similarly for $\delta^\vee$. For example, $n_\delta(\alpha) = q(\delta^\vee)^2 n_\alpha(\delta)$. But n_α is a W-invariant quadratic form on T^*, hence it is a scalar multiple of q. As $q(\alpha) = q(\delta^\vee)q(\delta)$, we have $n_\alpha(\alpha) = q(\delta^\vee)n_\alpha(\delta)$. But n_δ is also a scalar multiple of q, so we conclude that $n_\delta = q(\delta^\vee)n_\alpha$, proving the claim.

The *Dynkin index* $N(G_s)$ is defined to be the gcd of $N(\chi)$ as χ varies over the characters of finite-dimensional representations of G_s. The number $N(G_s)$ is calculated in [10], [18], or [19]:

type of G_s	A or C	B_n $(n \geq 3)$, D_n $(n \geq 4)$, G_2	F_4 or E_6	E_7	E_8
$N(G_s)$	1	2	6	12	60

If G is a simple and strongly inner group, then, for the purposes of this paper, we define the Dynkin index $N(G)$ of G to be the Dynkin index $N(G_s)$ of the split simply connected group of the same Killing-Cartan type.

2. Dynkin indices and the map ϕ

Let G_s denote a split simply connected simple linear algebraic group of rank n over a field k. We fix a pinning for G_s and in particular a split maximal torus T and fundamental weights $\omega_1,\ldots,\omega_n$. As G_s is simply connected, T_* $(= \mathrm{Hom}(\mathbb{G}_m, T))$ and T^* are canonically identified with the coroot and weight lattices respectively.

2.1. **Definition.** Put $\mathbb{Z}[T^*] := \mathbb{Z}[e^{\omega_1},\ldots,e^{\omega_n}]$, the integral group ring, and $\mathbb{S}(T^*) := \mathbb{Z}[\omega_1,\ldots,\omega_n]$, the symmetric algebra of T^*. We define a ring homomorphism

$$\phi_m\colon \mathbb{Z}[T^*]/\gamma^{m+1} \to \mathbb{S}(T^*)/(\mathbb{S}^{m+1}(T^*)),\ m\geq 2,$$

$$\text{via } \phi_m\left(e^{\sum_{i=1}^n a_i\omega_i}\right) = \prod_{i=1}^n (1-\omega_i)^{-a_i}.$$

In particular, $\phi_m(e^{\omega_i}) = 1+\omega_i+\cdots+\omega_i^m$ and $\phi_m(e^{-\omega_i}) = 1-\omega_i$. (Note that $\mathbb{Z}[T^*]$ can be viewed as Laurent polynomials in the variables $\omega_1,\ldots,\omega_n$, and from this perspective it is clear that the formula for ϕ gives a well-defined ring homomorphism on $\mathbb{Z}[T^*]$ and $\phi_m(\gamma^{m+1})$ is zero in $\mathbb{S}(T^*)/(\mathbb{S}^{m+1}(T^*))$.)

The homomorphism ϕ_m is an isomorphism. To see this, define a homomorphism $\mathbb{S}(T^*) \to \mathbb{Z}[T^*]/\gamma^{m+1}$ via $\psi_m(\omega_i) = 1 - e^{-\omega_i}$ for all i; it induces a homomorphism $\mathbb{S}(T^*)/(\mathbb{S}^{m+1}(T^*)) \to \mathbb{Z}[T^*]/\gamma^{m+1}$ that we also denote by ψ_m. As the compositions $\phi_m\psi_m$ and $\psi_m\phi_m$ are both the identity on generators, the claim is proved.

The goal of this section is to give a formula for ϕ_2 on I'.

2.2. **Proposition.** *If G_s is simple, then for $\chi\in\mathbb{Z}[T^*]^W$, we have:*

$$\phi_2(\chi) = \mathrm{aug}(\chi) + N(\chi)\cdot q \quad \in \left(\mathbb{S}(T^*)/(\mathbb{S}^3(T^*))\right)^W,$$

where q is the invariant form introduced in §§1B.

The proof would be much easier if we already knew that ϕ_2 takes W-invariant elements to W-invariant elements, but this only comes as a consequence of the proof of the proposition. We give some preliminary material before the proof.

2.3. **Example** (SL_2)**.** In case $G_s = \mathrm{SL}_2$, write ω for the unique fundamental weight. For $n > 0$, we have:

$$\phi_2(e^{n\omega} + e^{-n\omega}) = (1 + \omega + \omega^2)^n + (1 - \omega)^n = 2 + n^2\omega^2,$$

which verifies Prop. 2.2 for this group.

2.4. **Example** ($\mathrm{SL}_2 \times \mathrm{SL}_2$)**.** In case $G_s = \mathrm{SL}_2 \times \mathrm{SL}_2$ there are two fundamental weights ω_1, ω_2 and the Weyl group W is the Klein four-group; it acts by flipping the signs of ω_1 and ω_2. The definition of ϕ_2 above makes sense here even though G_s is not simple. We find:

$$\phi_2(We^{a_1\omega_1 + a_2\omega_2}) = 4 + 2\left[a_1^2\omega_1^2 + a_2^2\omega_2^2\right].$$

One final observation about Weyl group actions. We write $W\lambda$ for the W-orbit of $\lambda \in T^*$.

2.5. **Lemma.** *For every root α and weight $\lambda \in T^*$, the map $W\lambda \to \mathbb{Z}$ defined by $\pi \mapsto \langle \pi, \alpha^\vee \rangle$ hits x and $-x$ the same number of times, for every $x \in \mathbb{Z}$. If α, β are orthogonal roots, then for every weight $\lambda \in T^*$, the map $W\lambda \to \mathbb{Z} \times \mathbb{Z}$ defined by $\pi \mapsto (\langle \pi, \alpha^\vee \rangle, \langle \pi, \beta^\vee \rangle)$ hits (x, y), $(-x, y)$, $(x, -y)$, and $(-x, -y)$ the same number of times, for every $x, y \in \mathbb{Z}$.*

Sketch of proof. It is an exercise to show the analogous statements for the map $W \to \mathbb{Z}$ defined by $w \mapsto \langle w\lambda, \alpha^\vee \rangle$ and similarly for the second claim. The lemma follows. $\square$

Proof of Prop. 2.2. We may assume that $\chi = \sum e^{\lambda_j}$ where $\lambda_1, \dots, \lambda_r$ is the Weyl orbit of some $\lambda \in T^*$. Put $\lambda_j = \sum_{i=1}^n a_{ij}\omega_i$, so $\phi(\chi) = \sum_{j=1}^r \prod_{i=1}^n (1 - \omega_i)^{-a_{ij}}$. Obviously, the degree 0 component of $\phi(\chi)$ is $r = \mathrm{aug}(\chi)$.

The degree 1 component of $\phi(\chi)$ is $\sum_j \sum_i a_{ij}\omega_i = \sum_i \left(\sum_j a_{ij}\right)\omega_i$. Here the claim is that $\sum_j a_{ij} = 0$ for each i. The a_{ij}'s are the images of $W\lambda$ in $\mathbb{Z}$ under the map $\lambda_j \mapsto \langle \lambda_j, \alpha_i^\vee \rangle$ where α_i denotes the simple root corresponding to the fundamental weight ω_i, hence the claim follows from Lemma 2.5.

The crux is to check the claim on the degree 2 component q_1 of $\phi(\chi)$; it is an integer-valued quadratic form on the coroot lattice T_* and we check that it equals $q_2 := N(\chi)q$. We write out for $\ell = 1, 2$:

$$q_\ell\left(\sum d_i\alpha_i^\vee\right) = \sum_i d_i^2 q_\ell(\alpha_i^\vee) + \sum_{i<j} d_i d_j b_{q_\ell}(\alpha_i^\vee, \alpha_j^\vee), \tag{2.6}$$

where b_{q_ℓ} is the polar bilinear form of q_ℓ. We will check that the value of this expression is the same for $\ell = 1, 2$.

First suppose that $\delta^\vee := \sum d_i\alpha_i^\vee$ is a coroot and every d_i is 0 or 1. Then it defines a homomorphism $\eta\colon \mathrm{SL}_2 \to G_s$ so that, roughly speaking, the simple coroot $\alpha^\vee$ of SL_2 (viewed as a map $\mathbb{G}_m \to T_1 := \eta^{-1}(T)$) satisfies $\eta(\alpha^\vee) = \delta^\vee$. We check that the diagram

$$\begin{array}{ccc} \mathbb{Z}[T^*] & \xrightarrow{\phi_2} & \mathbb{S}(T^*)/(\mathbb{S}^3(T^*)) \\ {\scriptstyle\eta^*}\downarrow & & \downarrow{\scriptstyle\eta^*} \\ \mathbb{Z}[T_1^*] & \xrightarrow{\phi_2} & \mathbb{S}(T_1^*)/(\mathbb{S}^3(T_1^*)) \end{array} \tag{2.7}$$

commutes. Since $\omega_j(\delta^\vee) = d_j$, we have $\eta^*(\omega_j) = d_j\omega$ for ω the fundamental weight of SL_2 dual to $\alpha^\vee$. We find:

$$\eta^*\phi_2(e^{\sum c_j\omega_j}) = \prod\nolimits_j (1-d_j\omega)^{-c_j} = (1-\omega)^{-\sum c_j d_j},$$

because the d_j are all 0 or 1. As this is $\phi_2(e^{(\sum d_j c_j)\omega}) = \phi_2\eta^*(e^{\sum c_j\omega_j})$, we have confirmed the commutativity of (2.7).

Put ϕ^2 for the composition of ϕ_2 with the projection onto the degree 2 component $\mathbb{S}^2$, so $q_1 = \phi^2(\chi)$. Then $q_1(\delta^\vee) = (\eta^*\phi^2(\chi))(\alpha^\vee)$ obviously, which is $(\phi^2\eta^*(\chi))(\delta^\vee)$ by commutativity of (2.7). We have $\eta^*(\chi) = \sum_j e^{\sum_i a_{ij}\omega}$ and by Lemma 2.5, the multiset of the j integers $\sum_i a_{ij}d_i$ is symmetric under multiplication by -1, hence by Example 2.3 we find:

$$q_1(\delta^\vee) = \frac{1}{2}\left(\sum\nolimits_j \left(\sum\nolimits_i a_{ij}d_i\right)^2\right) = \frac{1}{2}\sum\nolimits_j \langle\lambda_j,\delta^\vee\rangle^2.$$

By (1.3) this equals $q(\delta^\vee)N(\chi) = q_2(\delta^\vee)$.

Returning to equation (2.6), this shows that the term $q_\ell(\alpha_i^\vee)$ does not depend on ℓ. Similarly, if $\alpha_i^\vee$ and $\alpha_j^\vee$ are not orthogonal coroots, then $\alpha_i^\vee$ and $\alpha_j^\vee$ are adjacent in the Dynkin diagram and $\alpha_i^\vee + \alpha_j^\vee$ is a coroot [2, VI.1.6, Cor. 3b]. The preceding two paragraphs show that the value of

$$b_{q_\ell}(\alpha_i^\vee,\alpha_j^\vee) = q_\ell(\alpha_i^\vee+\alpha_j^\vee) - q_\ell(\alpha_i^\vee) - q_\ell(\alpha_j^\vee)$$

does not depend on ℓ.

It remains to consider $b_{q_\ell}(\alpha_i^\vee,\alpha_j^\vee)$ where $\alpha_i^\vee$ and $\alpha_j^\vee$ are orthogonal (relative to the polar form of q – it follows that they are orthogonal relative to b_{q_2}. We use $\alpha_i^\vee$ and $\alpha_j^\vee$ to define a homomorphism $\tau\colon \mathrm{SL}_2\times\mathrm{SL}_2 \to G_s$ and – as we did for SL_2 above – we fix a torus $T_2 = T_1\times T_1 \subset \mathrm{SL}_2\times\mathrm{SL}_2$ such that $\tau(T_2) = \mathrm{im}(\alpha_i^\vee\times\alpha_j^\vee)\subset T$. Arguing using a commutative diagram analogous to (2.7), it suffices to check that the simple roots of $\mathrm{SL}_2\times\mathrm{SL}_2$ are orthogonal relative to $\tau q_1 = \phi^2\tau(\chi)$, which follows from Example 2.4 and Lemma 2.5. □

In view of §§1E, Prop. 2.2 gives:

2.8. **Corollary.** $\phi_2(I') = \mathbb{Z}\cdot N(G_s)\cdot q$. □

2.9. **Example.** Suppose G has type A_n for some $n\geq 2$. We continue the notation of §1C, and we compute:

$$\begin{aligned}\phi_3(We^{\omega_1}) &= \phi_3(e^{\omega_1} + e^{-\omega_n} + \sum_{i=2}^n e^{\omega_i-\omega_{i-1}})\\ &= (n+1) + q + \sum_{j=1}^n \omega_i^3 - \sum_{i=2}^n \omega_{i-1}\omega_i^2.\end{aligned}$$

The element $\Delta := We^{\omega_1} - We^{\omega_n}$ is in I', and by symmetry we see that $\phi_3(\Delta) = f_3$.

3. Torsion in the γ-filtration

Let $\mathfrak{B}$ denote the variety of Borel subgroups of a strongly inner simple linear algebraic group G over k. Recall that G is *strongly inner* if the simply connected cover of G is isomorphic to G_s twisted by a cocycle $\xi\in H^1_{\text{ét}}(k,G_s)$, where G_s denotes the simply connected split group of the same Killing-Cartan type as G. Observe that the variety $\mathfrak{B}$ is always defined over k by [7, Cor. XXVI.3.6]; it is a

twisted form of the variety of Borel subgroups $\mathfrak{B}_s$ of G_s, i.e., $\mathfrak{B}$ and $\mathfrak{B}_s$ become isomorphic over the algebraic closure of k.

In the present section we determine and bound respectively the torsion parts of the second and the third quotients of the γ-filtration on the variety $\mathfrak{B}$. The main result is the following.

3.1. **Theorem.** *Let $\mathfrak{B}$ be the variety of Borel subgroups of a strongly inner simple linear algebraic group G over a field k. Then*

- (i) $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ *is a cyclic group of order the Dynkin index $N(G)$ and generated by $\mathfrak{c}'(\theta)$ for θ as in Def. 3.3.*
- (ii) *The subgroup $\tau^3(\mathfrak{B})/\gamma^3(\mathfrak{B})$ of* $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ *is generated by $o(r(G))\,\mathfrak{c}'(\theta)$; it is cyclic of order $N(G)/o(r(G))$.*
- (iii) $2\operatorname{Tors}\gamma^{3/4}(\mathfrak{B})$ *is a quotient of $(\mathbb{Z}/N(G))^{\oplus(\operatorname{rank}G)}$.*

There is some new notation in the statement of the theorem. The *Rost invariant* r is a map $H^1(k,G_s)\to H^3(k,\mathbb{Q}/\mathbb{Z}(2))$, and for our G, the element $r(\xi)$ depends only on G and not on the choice of ξ by [11, Lemma 2.1]; we write simply $r(G)$ for this element and $o(r(G))$ for its order in the abelian group $H^3(k,\mathbb{Q}/\mathbb{Z}(2))$.

Philippe Gille pointed out to us that pasting together two results in the literature gives a description of $\operatorname{Tors}\operatorname{CH}^2(X)$ for some X.

3.2. **Proposition.** *Let X be a projective homogeneous variety under G. If G is split by $k(X)$, then* $\operatorname{Tors}\operatorname{CH}^2(X)$ *is a cyclic group whose order is the same as the order of $r(G)$ in $H^3(k,\mathbb{Q}/\mathbb{Z}(2))$; in particular its order divides $N(G)$.*

Proof. We view ξ as a principal homogeneous G_s-variety. The kernel of the scalar extension map $H^3(k,\mathbb{Q}/\mathbb{Z}(2))\to H^3(k(\xi),\mathbb{Q}/\mathbb{Z}(2))$ is the cyclic group generated by $r(G)$ by [10, p. 129]. For every extension L/k, ξ has an L-point if and only if G is split, if and only if X has an L-point. Therefore, this kernel is the same as the kernel of $H^3(k,\mathbb{Q}/\mathbb{Z}(2))\to H^3(k(X),\mathbb{Q}/\mathbb{Z}(2))$. A theorem of Peyre-Merkurjev [27] shows that this kernel is isomorphic to $\operatorname{Tors}\operatorname{CH}^2(X)$. $\square$

Obviously, one can take $X=\mathfrak{B}$ in the proposition. Furthermore, the same proof shows that the proposition still holds if one replaces "G is strongly inner" and "G is split by $k(X)$" with "G has trivial Tits algebras" and "G becomes quasi-split over $k(X)$".

Also, statement 3.1(i) makes use of the following definition.

3.3. **Definition.** Write the form q from subsection §1B (relative to the split group G_s) as $q=\sum_{i\le j}c_{ij}\omega_i\omega_j\in\mathbb{S}(T^*)$. We call the element

$$\theta:=\sum_{i\le j}c_{ij}(1-e^{-\omega_i})(1-e^{-\omega_j})\quad\in\mathbb{Z}[T^*]$$

the *special cycle*. Its image in $\mathbb{Z}[T^*]/\gamma^{m+1}$ is $\psi_m(q)$ for all $m\ge 2$.

Proof of Theorem 3.1. By the result of Panin [25, Thm. 2.2.(2)] since G is strongly inner, the restriction map

$$\operatorname{res}\colon K_0(\mathfrak{B})\to K_0(\mathfrak{B}\times k_{\mathrm{alg}})\simeq K_0(\mathfrak{B}_s\times k_{\mathrm{alg}})\simeq K_0(\mathfrak{B}_s)\tag{3.4}$$

is an isomorphism, where k_{alg} denotes an algebraic closure of k. Since the γ-filtration is defined in terms of Chern classes and the latter commute with restrictions, it induces an isomorphism between the γ-quotients, i.e.,

$$\operatorname{res}\colon\gamma^{i/i+1}(\mathfrak{B})\xrightarrow{\simeq}\gamma^{i/i+1}(\mathfrak{B}_s).$$

Therefore, we may reduce to the split case $G = G_s$. Let T, T^*, etc., be as in subsection §1B.

There is a commutative diagram

$$\begin{tikzcd}
\gamma^{m/m+1}(\mathfrak{B}_s) \arrow[rr, "c_m"] & & \mathrm{CH}^m(\mathfrak{B}_s) \\
\gamma^m/\gamma^{m+1} \arrow[u, two heads, "\mathfrak{c}'"] \arrow[rr, "(-1)^{m-1}(m-1)!\cdot\phi_m"] & & \mathbb{S}^m(T^*)/(\mathbb{S}^{m+1}(T^*)) \arrow[u, "\mathfrak{c}"']
\end{tikzcd} \tag{3.5}$$

First take $m = 2$ and suppose that $x \in \gamma^{2/3}$ maps to a torsion element in $\gamma^{2/3}(\mathfrak{B}_s)$. As $\mathrm{CH}^2(\mathfrak{B}_s)$ has zero torsion, the commutativity of (3.5) shows that $\phi_2(x)$ is in the kernel I of $\mathfrak{c}$. Writing $x = \sum_{i,j} a_{ij}(1 - e^{\omega_i})(1 - e^{\omega_j}) \mod \gamma^3$, we have $\phi_2(x) = \sum a_{ij}\omega_i\omega_j$ of degree 2 in I, hence $\phi_2(x) = aq$ for some $a \in \mathbb{Z}$. Then modulo γ^3, we have $x \equiv \psi_2\phi_2(x) \equiv a\theta$, so $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_s)$ is a cyclic group generated by the class of the special cycle θ modulo $\gamma^3 + I'$.

By Corollary 2.8 there exists $\chi \in I'$ such that $\phi_2(\chi) = N(G_s) \cdot q$. Applying ψ_2 we obtain that

$$0 \equiv \chi \equiv N(G_s) \cdot \theta \mod \gamma^3 + I',$$

hence, the order of θ modulo $\gamma^3 + I'$ divides the Dynkin index $N(G_s)$. This shows that $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ is a cyclic group of order dividing $N(G)$ with generator $\mathfrak{c}'(\theta)$.

Let $\xi' \in H^1(k', G_s)$ be a versal G_s-torsor for some extension k' of k, and write $\mathfrak{B}'$ for the Borel variety (over k') of the group G_s twisted by ξ'. The element $r(\xi')$ has order $N(G_s)$ in $H^3(k', \mathbb{Q}/\mathbb{Z}(2))$ by [10, pp. 31, 133]. But $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B}')$ is cyclic of order dividing $N(G_s)$, hence Prop. 3.2 and the exactness of (1.1) give that $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B}')$ also has order $N(G_s)$. Now take K to be an algebraically closed field containing k'. The restriction maps for $k \to K$ and $k' \to K$ give isomorphisms $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_{/k}) \simeq \operatorname{Tors}\gamma^{2/3}((\mathfrak{B}_s)_{/K}) \simeq \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}'_{/k'})$, which is itself $\mathbb{Z}/N(G)$, completing the proof of (i). Claim (ii) follows from the exactness of (1.1).

Now take $m = 3$ and suppose that $x \in \gamma^{3/4}$ maps to a torsion element in $\gamma^{3/4}(\mathfrak{B}_s)$. As $\mathrm{CH}^3(\mathfrak{B}_s)$ has zero torsion, diagram (3.5) shows that $2\phi_3(x)$ is in the kernel I of $\mathfrak{c}$. As in the $m = 2$ case, $2\phi_3(x)$ has degree 3.

Suppose G_s is not of type A_n for $n \geq 2$. Then by §§1C, $2\phi_3(x) = q \cdot f$, where $f = \sum_{i=1}^n a_i\omega_i$. Applying ψ_3 we obtain that $2x = \theta \cdot f'$, where $f' = \sum_{i=1}^n a_i(1 - e^{-\omega_i})$. In other words, the torsion part of $2\gamma^{3/4}(\mathfrak{B}_s)$ is generated by the elements $\mathfrak{c}'(\theta \cdot (1 - e^{-\omega_i}))$ for $i = 1, \dots, n$.

By Corollary 2.8 there exists $\chi \in I'$ such that $\phi_3(\chi \cdot (1 - e^{-\omega_i})) \equiv N(G_s) \cdot q \cdot \omega_i$ mod $(\mathbb{S}^4(T^*))$. Applying ψ_3 we obtain that

$$0 \equiv \chi \cdot (1 - e^{-\omega_i}) \equiv N(G_s) \cdot \theta \cdot (1 - e^{-\omega_i}) \mod \gamma^4 + I',$$

hence, the torsion part of $2\gamma^{3/4}(\mathfrak{B}_s)$ is a product of n cyclic groups of orders dividing the $N(G_s)$.

It remains to consider the case $m = 3$ and G_s of type A_n for $n \geq 2$; the claim is that $2x$ is in I'. As in the preceding paragraph, §§1C and Example 2.9 show that $2\phi_3(x) = q \cdot f + b\phi_3(\Delta)$ for some $b \in \mathbb{Z}$. Applying ψ_3, we find $2x = \theta \cdot f' + \Delta \cdot b$. As θ and Δ belong to I', the proof of (iii) is complete. $\square$

3.6. **Remark.** Observe that in the proof we used the structure of the ideal of invariants I in degrees 2 and 3. In principle it is possible to extend the proof to higher degrees, but then one must extend the arguments of §§1C and Example 2.9.

The next lemma can be used to extend the bounds obtained in Th. 3.1 to the case of a semisimple group.

3.7. **Lemma.** *Let $G_1, \dots, G_m$ be simple and strongly inner groups and write $\mathfrak{B}_j$ for the Borel variety of G_j. The Borel variety for $\prod G_j$ is $\prod \mathfrak{B}_j$, and we have:*

$$\operatorname{Tors}\gamma^{2/3}(\prod \mathfrak{B}_j) \simeq \bigoplus \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_j),$$

and

$$\operatorname{Tors}\gamma^{3/4}(\prod \mathfrak{B}_j) \simeq \bigoplus_{j=1}^{m} \big(\operatorname{Tors}\gamma^{3/4}(\mathfrak{B}_j) \oplus \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_j)\big).$$

Proof. Apply the Künneth decomposition and the fact that $\gamma^{i/i+1}(\mathfrak{B}_j)$ has no torsion for $i = 0$ and 1. □

4. Examples of torsion in CH^2

We now make a few remarks regarding torsion in CH^2. We maintain the notation of the previous section.

Using Prop. 3.2, one can view the the assignment $G \mapsto \operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})$ as a replacement for the Rost invariant of G. Furthermore, the group $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})$ is generated by the image of the special cycle $\theta \in \mathbb{Z}[T^*]$ which is defined in purely combinatorial terms.

The next two examples show that the image of θ generates the torsion in CH^2 for certain (generalized) Rost motives.

4.1. **Example.** Assume that the motive of $\mathfrak{B}$ splits as a direct sum of twisted copies of the Rost motive $\mathcal{R}_2$ corresponding to a 3-fold Pfister form. According to [26, §7] this can happen for G of type B_n, D_n, G_2, F_4, or E_6. Then we have

$$(\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})) \otimes \mathbb{Z}/2\mathbb{Z} \simeq \mathrm{Ch}^2(\mathcal{R}_2),$$

where Ch denotes the Chow group with $\mathbb{Z}/2\mathbb{Z}$-coefficients and the image of the special cycle θ generates $(\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})) \otimes \mathbb{Z}/2\mathbb{Z}$ and, hence, the group $\mathrm{Ch}(\mathcal{R}_2)$.

4.2. **Example.** Suppose G is an anisotropic group of type F_4 over k that is split by a cubic extension of k. (Such a group exists if and only if $H^3(k, \mathbb{Z}/3\mathbb{Z}(2))$ is not zero.) By [24, Rem. 4.5] and the main theorem of [26] the motive of the Borel variety $\mathfrak{B}$ splits as a direct sum of generalized Rost motives $\mathcal{R}_3$ corresponding to the Rost-Serre invariant g_3 (the mod-3 portion of the Rost invariant) of G. Therefore, we have

$$(\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})) \otimes \mathbb{Z}/3\mathbb{Z} \simeq \mathrm{Ch}^2(\mathcal{R}_3),$$

where Ch denotes the Chow group with $\mathbb{Z}/3\mathbb{Z}$-coefficients.

By the recent results of Merkurjev-Suslin [23] and Yagita [34, Thm. 10.5, Cor. 10.8] we have $\mathrm{Ch}^2(\mathcal{R}_3) \simeq \mathbb{Z}/3\mathbb{Z}$. On the other hand, the image of the special cycle θ generates $(\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B})) \otimes \mathbb{Z}/3\mathbb{Z}$ and, hence, the group $\mathrm{Ch}^2(\mathcal{R}_3)$.

5. Torsion in CH^3

Let X be a projective homogeneous G-variety such that G is split over $k(X)$. ("X is generically split.") Thanks to Proposition 3.2, we may view $\operatorname{Tors}\mathrm{CH}^2(X)$ as known, so we now investigate $\operatorname{Tors}\mathrm{CH}^3(X)$. We retain the meaning of G and $\mathfrak{B}$ from the previous section. Let n denote the rank of G and let r denote the rank of the Picard group of X over an algebraic closure of k.

We remark that the results in this section only use the fact that $\operatorname{Tors}\mathrm{CH}^2(X)$ is cyclic of order dividing $N(G)$, which follows from our Theorem 3.1(i) and Eq. (1.1). They do not need the finer result of Prop. 3.2, hence also do not need material from [10] and [27].

For an abelian group A and a prime p, write $\operatorname{Tors}_p A$ for the subgroup of A consisting of elements of order a power of p.

5.1. **Lemma.** *The restriction of the m-th Chern class gives a surjection*

$$\operatorname{Tors}\tau^{m/m+1}(X) \twoheadrightarrow (m-1)!\operatorname{Tors}\mathrm{CH}^m(X)$$

and for each prime p not dividing $(m-1)!$, c_m is an isomorphism

$$\operatorname{Tors}_p\tau^{m/m+1}(X) \xrightarrow{\simeq} \operatorname{Tors}_p\mathrm{CH}^m(X).$$

Proof. By Riemann-Roch (see subsection §1A), the composition

$$\mathrm{CH}^m(X) \overset{\mathrm{pr}}{\twoheadrightarrow} \tau^{m/m+1}(X) \xrightarrow{c_m} \mathrm{CH}^m(X)$$

is multiplication by $(-1)^{m-1}(m-1)!$, hence $c_m(\tau^{m/m+1}(X))$ is $(m-1)!\,\mathrm{CH}^m(X)$. For $x \in \operatorname{Tors}\mathrm{CH}^m(X)$, we have $(m-1)!\cdot x = c_m(\mathrm{pr}(x))$, where $\mathrm{pr}(x)$ is in $\operatorname{Tors}\tau^{m/m+1}(X)$. This proves the first claim, from which the second claim follows immediately. $\square$

5.2. **Proposition.** *If $\tau^3(\mathfrak{B}) = \gamma^3(\mathfrak{B})$, then $\operatorname{Tors} 4\cdot\mathrm{CH}^3(\mathfrak{B})$ is a quotient of the direct sum $(\mathbb{Z}/N(G)\mathbb{Z})^{\oplus n}$. In particular, the torsion part of $\mathrm{CH}^3(\mathfrak{B})$ can consist only of subgroups $\mathbb{Z}/2^s\mathbb{Z}$ for $s \le 4$, $\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/5\mathbb{Z}$.*

Theorem 3.1(ii) gives a way to check the hypothesis on the filtration.

Proof. By the hypothesis, the map $\gamma^{3/4}(\mathfrak{B}) \to \tau^{3/4}(\mathfrak{B})$ is surjective. Now combine Lemma 5.1 and Theorem 3.1(iii). $\square$

As an alternative to making a hypothesis on the filtrations, we may control the torsion on $\mathrm{CH}^3(X)$ based on information about the torsion in $\mathrm{CH}^2(X)$ and the motivic decomposition of X, as we now illustrate.

Fix a prime p. In the category of Chow motives with $\mathbb{Z}/p\mathbb{Z}$-coefficients, the motive of X is a direct sum of shifts of an indecomposable motive $\mathcal{R}$, see [26, Th. 5.17], where $\mathcal{R}$ depends on G but not the choice of X (ibid., Th. 3.7). We write $\mathrm{Ch}^m(\mathcal{R})$ for the m-th Chow group of $\mathcal{R}$ with $\mathbb{Z}/p\mathbb{Z}$ coefficients.

5.3. **Lemma.** *We have:*

(i) $(\operatorname{Tors}_p\mathrm{CH}^2(X))\otimes\mathbb{Z}/p\mathbb{Z} \simeq \mathrm{Ch}^2(\mathcal{R})$*;*
(ii) $(\operatorname{Tors}_p\mathrm{CH}^3(X))\otimes\mathbb{Z}/p\mathbb{Z} \simeq (\mathrm{Ch}^2(\mathcal{R}))^{\oplus r}\oplus\mathrm{Ch}^3(\mathcal{R})$.
(iii) $\operatorname{Tors}\mathrm{CH}^3(\mathfrak{B}) \simeq (\operatorname{Tors}\mathrm{CH}^2(X))^{\oplus(n-r)}\oplus\operatorname{Tors}\mathrm{CH}^3(X)$.

Proof. The expression of the motive of X from [26] gives:

$$(\mathrm{Tors}_p \,\mathrm{CH}^m(X)) \otimes \mathbb{Z}/p\mathbb{Z} \simeq \overline{\mathrm{Ch}}^m(\mathcal{R}) \oplus (\overline{\mathrm{Ch}}^{m-1}(R))^{\oplus r} \oplus (\overline{\mathrm{Ch}}^{m-2}(\mathcal{R}))^{\oplus \cdots} \oplus \ldots$$

where $\overline{\mathrm{Ch}}^m(\mathcal{R})$ denotes the kernel of the restriction $\mathrm{Ch}^m(\mathcal{R}) \to \mathrm{Ch}^m(\mathcal{R} \times_k \bar{k})$ to the algebraic closure $\bar{k}$. By the formula for the generating function [26, Thm. 5.13(3)] and table 4.13 in ibid., we have $\overline{\mathrm{Ch}}^0(\mathcal{R}) = \overline{\mathrm{Ch}}^1(\mathcal{R}) = 0$ and $\overline{\mathrm{Ch}}^i(\mathcal{R}) = \mathrm{Ch}^i(\mathcal{R})$ for $i = 2, 3$. This implies claims (i) and (ii).

Claim (iii) is proved similarly, but using the integral motivic decomposition from [26, Th. 3.7] with $Y = \mathfrak{B}$. □

5.4. **Proposition.** *Fix an odd prime p. If $\mathrm{Tors}_p \,\mathrm{CH}^2(X) \neq 0$, then*

1. $p = 3$ *or* 5*;*
2. $\mathrm{Ch}^2(\mathcal{R}) \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Ch}^3(\mathcal{R}) = 0$.
3. $\mathrm{Tors}_p \,\mathrm{CH}^2(X) \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $\mathrm{Tors}_p \,\mathrm{CH}^3(X) \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus r}$.

Proof. By Prop. 3.2 (or [26, Th. 3.7]), $\mathrm{CH}^2(X)$ and $\mathrm{CH}^2(\mathfrak{B})$ have the same p-torsion. As $\mathrm{Tors}\,\mathrm{CH}^2(\mathfrak{B})$ has order dividing $N(G)$ by Theorem 3.1(i), the list of Dynkin indexes in §1E gives that $p = 3$ or 5 and $\mathrm{Tors}_p \,\mathrm{CH}^2(\mathfrak{B}) \simeq \mathbb{Z}/p\mathbb{Z}$. Combining this with Lemma 5.3(i), it only remains to prove the claims about $\mathrm{Ch}^3(\mathcal{R})$ and $\mathrm{CH}^3(X)$.

Tensoring sequence (1.1) with $\mathbb{Z}/p\mathbb{Z}$, we find that

$$\gamma^3(\mathfrak{B}) \otimes \mathbb{Z}/p\mathbb{Z} = \tau^3(\mathfrak{B}) \otimes \mathbb{Z}/p\mathbb{Z}.$$

Combining Lemma 5.1 and Theorem 3.1(iii) gives that $\mathrm{Tors}_p \,\mathrm{CH}^3(\mathfrak{B})$ is a product of at most n copies of $\mathbb{Z}/p\mathbb{Z}$. By Lemma 5.3(ii) applied to $X = \mathfrak{B}$ we obtain

$$(\mathrm{Tors}_p \,\mathrm{CH}^3(\mathfrak{B})) \otimes \mathbb{Z}/p\mathbb{Z} \simeq (\mathbb{Z}/p\mathbb{Z})^{\oplus n} \oplus \mathrm{Ch}^3(\mathcal{R}).$$

Since the right hand side already contains n copies of $\mathbb{Z}/p\mathbb{Z}$, $\mathrm{Tors}_p \,\mathrm{CH}^3(\mathfrak{B}) = (\mathbb{Z}/p\mathbb{Z})^{\oplus n}$ and $\mathrm{Ch}^3(\mathcal{R})$ is zero. The second part of (3) now follows by Lemma 5.3(iii). □

6. Cohomological invariants and the Tits algebras

So far, we have studied the case where G is strongly inner and we constructed the special cocycle $\mathfrak{c}'(\theta)$ in $K_0(\mathfrak{B})$, cf. Example 6.3 below. We now relax our hypothesis on G and ask if $\mathfrak{c}'(\theta)$ is still defined over k.

In the present section G_s denotes an adjoint split simple linear algebraic group over a field k. As it is adjoint, the character group T^* of a split maximal torus of G_s is naturally identified with the root lattice Λ_r.

We fix a pinning for G_s, which includes a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ in Λ_r. Write ω_i for the fundamental weight corresponding to α_i and s_i for the reflection of the weight lattice Λ in the hyperplane orthogonal to α.

The Steinberg basis. For each element w of the Weyl group W (of T) we define

$$\rho_w := \sum_{\{i \in 1 \ldots n \mid w^{-1}(\alpha_i) < 0\}} w^{-1}(\omega_i) \quad \in \Lambda.$$

Let $\mathbb{Z}[\Lambda]^W$ denote the subring of W-invariant elements. Then the integral group ring $\mathbb{Z}[\Lambda]$ is a free $\mathbb{Z}[\Lambda]^W$-module with the basis $\{e^{\rho_w} \mid w \in W\}$ by [32, Th. 2.2].

Let $\mathfrak{B}_s$ denote the variety of Borel subgroups of G_s. Consider the characteristic map for the simply connected cover of G_s

$$\mathfrak{c}'\colon \mathbb{Z}[\Lambda] \twoheadrightarrow K_0(\mathfrak{B}_s).$$

Since the kernel of the surjection $\mathfrak{c}'$ is generated by elements $x \in \mathbb{Z}[\Lambda]^W$ in the kernel of the augmentation map, there is an isomorphism

$$\mathbb{Z}[\Lambda] \otimes_{\mathbb{Z}[\Lambda]^W} \mathbb{Z} \simeq \mathbb{Z}[\Lambda]/\ker(\mathfrak{c}') \simeq K_0(\mathfrak{B}_s).$$

The elements

$$\{g_w := \mathfrak{c}'(e^{\rho_w}) = [\mathcal{L}(\rho_w)] \mid w \in W\}$$

form a free $\mathbb{Z}$-basis of $K_0(\mathfrak{B}_s)$ called the *Steinberg basis*.

Observe that the quotient group Λ/Λ_r coincides with the group of characters of the center of the simply connected cover of G_s. Consider the surjective ring homomorphism induced by the restriction $\mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda/\Lambda_r]$. Since W acts trivially on Λ/Λ_r, we obtain that

$$\bar{\rho}_w = \sum_{\{i\in 1\ldots n \mid w^{-1}(\alpha_i)<0\}} \bar{\omega}_i,$$

where $\bar{\rho}$ means the restriction to Λ/Λ_r.

6.1. **Example.** (a) For a simple reflection s_j we have

$$\rho_{s_j} = \sum_{\{i\in 1\ldots n \mid s_j(\alpha_i)<0\}} s_j(\omega_i) = s_j(\omega_j) = \omega_j - \alpha_j.$$

(b) More generally, let $w := s_{i_1}s_{i_2}\ldots s_{i_m}$ be a product of m distinct simple reflections such that the simple roots $\alpha_{i_j}, \alpha_{i_\ell}$ are orthogonal for all $j \neq \ell$. Then

$$\rho_{s_{i_1}s_{i_2}\ldots s_{i_m}} = \rho_{s_{i_1}} + \rho_{s_{i_2}} + \ldots + \rho_{s_{i_m}}.$$

because $w^{-1}(\alpha_i)$ is negative if and only if $i = i_j$ for some j.

(c) For a product of two simple reflections s_is_j such that $c_{ij} = \alpha_i^\vee(\alpha_j) < 0$ we obtain

$$\rho_{s_is_j} = \rho_{s_i} + c_{ij}\alpha_j.$$

The Tits algebras and the base change. Let G be a twisted form of G_s, i.e. G is obtained by twisting G_s by a cocycle $\xi \in Z^1(k, \mathrm{Aut}(G_s))$. More specifically, our choice of pinning for G_s defines a section s of the quotient map $\pi\colon \mathrm{Aut}(G_s) \to \mathrm{Aut}(\Delta)$. Twisting G_s by $\xi' := s\pi(\xi)$ gives a quasi-split group G_q and we pick $\xi'' \in Z^1(k, G_q)$ that maps via twisting to ξ—i.e., we pick ξ'' so that G is isomorphic to ${}_{\xi''}G_q$.

Let $\mathfrak{B} = {}_\xi\mathfrak{B}_s$ be the variety of Borel subgroups of G. Let Γ denote the absolute Galois group of k; it acts on the weight lattice Λ via the cocycle ξ'.

Following [33] (see also [25, §3.1, §11.7] and [22, §2]) we associate with each $\chi \in \Lambda/\Lambda_r$ the field of definition k_χ of χ and the central simple algebra $A_{\chi,\xi}$ over k_χ called the Tits algebra. Here k_χ is a fixed subfield for the stabilizer

$$\Gamma_\chi = \{\tau \in \Gamma \mid \tau(\chi) = \chi\}.$$

There is a group homomorphism

$$\beta\colon (\Lambda/\Lambda_r)^{\Gamma_\chi} \to \mathrm{Br}(k_\chi) \text{ with } \beta(\chi') = [A_{\chi',\xi}].$$

By [27, Thm. 2.1] there is an isomorphism

$$\operatorname{Tors}\operatorname{CH}^2(\mathfrak{B}) \simeq \frac{\ker\big(H^3(k,\mathbb{Q}/\mathbb{Z}(2)) \to H^3(k(\mathfrak{B}),\mathbb{Q}/\mathbb{Z}(2))\big)}{\bigoplus_{\chi\in\Lambda/\Lambda_r} N_{k_\chi/k}\big(k_\chi^* \cup \beta(\chi)\big)},$$

where the numerator is the kernel of the restriction map to the field of fractions $k(\mathfrak{B})$ of $\mathfrak{B}$ and $N_{k_\chi/k}$ is the norm map. Let $H^3_\beta(k,\mathbb{Q}/\mathbb{Z}(2))$ denote the cohomology quotient

$$H^3_\beta(k,\mathbb{Q}/\mathbb{Z}(2)) = H^3(k,\mathbb{Q}/\mathbb{Z}(2))/\bigoplus_{\chi\in\Lambda/\Lambda_r} N_{k_\chi/k}\big(k_\chi^*\cup\beta(\chi)\big)$$

so that $\operatorname{Tors}\operatorname{CH}^2(\mathfrak{B}) \subseteq H^3_\beta(k,\mathbb{Q}/\mathbb{Z}(2))$.

Let l/k be a field extension. Since the Chern classes commute with restrictions, there is the induced map

$$\operatorname{res}_{l/k}\colon \gamma^{i/i+1}(\mathfrak{B}) \to \gamma^{i/i+1}(\mathfrak{B}_l),$$

where $\mathfrak{B}_l = \mathfrak{B}\times_k l$, with the image generated by the products

$$\langle c^{K_0}_{n_1}(x_1)\cdots c^{K_0}_{n_m}(x_m) \mid n_1+\cdots+n_m = i,\ x_1,\ldots,x_m \in \operatorname{res}_{l/k}\big(K_0(\mathfrak{B})\big)\rangle$$

and there is a commutative diagram

$$\begin{tikzcd} \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}) \arrow[r, two heads, "c_2"] \arrow[d, "\operatorname{res}_{l/k}"'] & \operatorname{Tors}\operatorname{CH}^2(\mathfrak{B}) \subseteq H^3_\beta(k,\mathbb{Q}/\mathbb{Z}(2)) \arrow[d, "\operatorname{res}_{l/k}"] \\ \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_l) \arrow[r, two heads, "c_2"] & \operatorname{Tors}\operatorname{CH}^2(\mathfrak{B}_l) \subseteq H^3_\beta(l,\mathbb{Q}/\mathbb{Z}(2)) \end{tikzcd} \tag{6.2}$$

Observe that the image $\operatorname{res}_{l/k}\big(K_0(\mathfrak{B})\big)$ can be computed using [25]. For instance, if G is an inner group, i.e., $\xi' = 0$, then Γ acts trivially on Λ/Λ_r, i.e. $k_\chi = k$ for all χ and by [25, Thm. 4.2] the image of the restriction map $K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$ from (3.4) coincides with the sublattice

$$\langle \operatorname{ind}(A_{\bar\rho_w,\xi})\cdot g_w \mid w\in W\rangle.$$

Using (6.2) one can provide a non-trivial element in $H^3_\beta(k,\mathbb{Q}/\mathbb{Z}(2))$ as follows:

- Assume that we are given non-trivial elements over l, i.e. that there is a non-trivial element $\theta \in \operatorname{Tors}\gamma^{2/3}(\mathfrak{B}_l)$ such that $c_2(\theta) \in H^3_\beta(l,\mathbb{Q}/\mathbb{Z}(2))$ is non-trivial.
- Assume that we know that θ is defined over k, i.e. that $\theta = \operatorname{res}_{l/k}(\theta')$ for some $\theta' \in \operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$.

Then the image $c_2(\theta')$ provides a non-trivial element in $H^3_\beta(k,\mathbb{Q}/\mathbb{Z}(2))$.

6.3. **Example** (strongly inner case)**.** If G is strongly inner—i.e., if G is inner and β is the trivial homomorphism—then for any field extension l/k the left vertical arrow in (6.2) is an isomorphism, hence, identifying $\operatorname{Tors}\gamma^{2/3}(\mathfrak{B})$ with the cyclic group generated by the special cycle θ. As in Prop. 3.2 and its proof $\operatorname{Tors}\operatorname{CH}^2(\mathfrak{B})$ coincides with the usual unramified cohomology generated by the Rost invariant $r(G)$ of G and $\langle c_2(\theta)\rangle = \langle r(G)\rangle$ in $H^3(k,\mathbb{Q}/\mathbb{Z}(2))$.

6.4. **Lemma.** *Assume that G is inner.*

(a) *If a weight ω is such that $\beta(\omega)=0$, then $[\mathcal{L}(\omega)]$ is in the image of*

$$\mathrm{res}\colon K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s).$$

In particular, it holds for the classes $[\mathcal{L}(\alpha_i)]$ of simple roots α_i.

Under the notation of Example 6.1(b) we have

(b) $\sum_j c_1^{K_0}([\mathcal{L}(\omega_{i_j})]) - c_1^{K_0}([\mathcal{L}(\alpha_{i_j})]) \equiv c_1^{K_0}\left(\prod_j g_{s_{i_j}}\right) \equiv c_1^{K_0}(g_w) \mod \gamma^3(\mathfrak{B}_s)$;

(c) *If $\beta(\sum_j \omega_{i_j})=0$, then $\sum_j c_1^{K_0}([\mathcal{L}(\omega_{i_j})])$ is in the image of*

$$\mathrm{res}\colon \gamma^{1/2}(\mathfrak{B}) \to \gamma^{1/2}(\mathfrak{B}_s).$$

Proof. (a) follows by [14, Cor. 3.1]. (b) follows by the formula for the first Chern class (in K_0) of the tensor product of line bundles. According to (a) each $c_1^{K_0}([\mathcal{L}(\alpha_{i_j})])$ is in the image of the restriction map which implies (c). □

6.5. **Proposition** (quaternionic inner case)**.** *Assume that G is inner. If every Tits algebra of ${}_\xi G_s$ has index 1 or 2, then the special cycle θ is in the image of the restriction map*

$$\mathrm{res}\colon \gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_s).$$

In other words, if l/k is an extension that kills $\operatorname{im}\beta$, then the image of c_2 over l coincides with the subgroup generated by the respective Rost invariant, i.e. we have

$$\mathrm{im}(c_2)_l = \langle r(G_l)\rangle \subseteq H^3(l,\mathbb{Q}/\mathbb{Z}(2)).$$

Proof of Prop. 6.5. We may assume that $N(G)$ is not 1 (otherwise θ maps to zero in $\gamma^{2/3}(\mathfrak{B}_s)$ by Th. 3.1) and Λ/Λ_r has even order (otherwise Example 6.3 applies), i.e., we may assume that G has type B, C, D, or E_7.

We first make a general observation. Mod $\gamma^3(\mathfrak{B}_s)$, we have:

$$\begin{aligned} c_1^{K_0}([\mathcal{L}(\omega_i)])^2 &\equiv (c_1^{K_0}(g_{s_i}) + c_1^{K_0}([\mathcal{L}(\alpha_i)]))^2 \\ &\equiv c_1^{K_0}(g_{s_i})^2 + 2c_1^{K_0}(g_{s_i})c_1^{K_0}([\mathcal{L}(\alpha_i)]) + c_1^{K_0}([\mathcal{L}(\alpha_i)])^2. \end{aligned}$$

The Whitney Sum Formula gives that $c_2^{K_0}(2g_{s_i}) = c_1^{K_0}(g_{s_i})^2$ and $c_1^{K_0}(2g_{s_i}) \equiv 2c_1^{K_0}(g_{s_i}) \mod \gamma^2(\mathfrak{B}_s)$. Our hypothesis on the Tits algebras gives that $2g_{s_i}$ is in the image of $K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$, and it follows that $c_1^{K_0}([\mathcal{L}(\omega_i)])^2$ is *rational* – i.e., is in the image of $\gamma^{2/3}(\mathfrak{B}) \to \gamma^{2/3}(\mathfrak{B}_s)$ – for all i.

<u>*Type E_7*</u>: Suppose that G has type E_7. Then

$$q = \left(\sum_{i=1}^{7} \omega_i^2\right) - \omega_1\omega_3 - \omega_3\omega_4 - \omega_4\omega_2 - \omega_4\omega_5 - \omega_5\omega_6 - \omega_6\omega_7$$

where we have numbered the roots following [2]. Each ω_i^2 contributes a term of the form $c_1^{K_0}([\mathcal{L}(\omega_i)])^2$ to the image of $\mathfrak{c}'(\theta)$ in $\gamma^{2/3}(\mathfrak{B}_s)$, and such a term is rational by the preceding paragraph. The weights $\omega_1,\omega_3,\omega_4,\omega_6$ belong to the root lattice and so the term $\omega_1\omega_3$ contributes a rational term $c_1^{K_0}([\mathcal{L}(\omega_1)])c_1^{K_0}([\mathcal{L}(\omega_3)])$ to $\mathfrak{c}'(\theta)$, and similarly for the term $\omega_3\omega_4$. Next we observe that $\omega_4\omega_2+\omega_4\omega_5$ contributes

$$c_1^{K_0}([\mathcal{L}(\omega_4)])\left(c_1^{K_0}([\mathcal{L}(\omega_2)]) + c_1^{K_0}([\mathcal{L}(\omega_5)])\right)$$

to $\mathfrak{c}'(\theta)$. But ω_4 and $\omega_2+\omega_5$ both lie in the root lattice, so both terms in the product are rational by Lemma 6.4. The same argument handles $\omega_5\omega_6+\omega_6\omega_7$, and we are done with the E_7 case.

Type D: Suppose that G has type D_n. Then

$$q = \sum_{i=1}^{n} \omega_i^2 - \sum_{i=1}^{n-2} \omega_i\omega_{i+1} - \omega_{n-2}\omega_n.$$

The terms ω_i^2 are treated as in the E_7 case. For the terms in the second sum, we collect around terms with even subscripts: for even $i < n-2$, consider $\omega_i(\omega_{i-1} + \omega_{i+1})$. As ω_i and $\omega_{i-1} + \omega_{i+1}$ belong to the root lattice, we see as in the E_7 case that they contribute rational terms to $\mathfrak{c}'(\theta)$.

Suppose now that n is even. Then we have not accounted for $\omega_{n-2}(\omega_{n-3} + \omega_{n-1} + \omega_n)$ from q. As both terms in the product belong to the root lattice, we are finished as in the E_7 case.

If n is odd, then we have not accounted for $\omega_{n-2}(\omega_{n-1} + \omega_n)$ in q. Here Λ/Λ_r is isomorphic to $\mathbb{Z}/4$ and $\omega_{n-2}, \omega_{n-1}, \omega_n$ map to $2, \pm1, \pm3$ respectively. In particular, $\beta(\omega_{n-2}) = 2\beta(\omega_n)$, which is zero by our hypothesis on the Tits algebras, so $[\mathcal{L}(\omega_{n-2})]$ is in the image of $\operatorname{res}\colon K_0(\mathfrak{B}) \to K_0(\mathfrak{B}_s)$. Similarly, $\beta(\omega_{n-1} + \omega_n) = \beta(\omega_{n-1}) + \beta(\omega_n) = 0$, and as in the E_7 case, we see that $\mathfrak{c}'(\theta)$ is rational.

Type B or C: If G has type B_n or C_n, Λ/Λ_r equals $\mathbb{Z}/2$. In either case,

$$q = \sum_{i=1}^{n} c_{ii}\omega_i^2 - \sum_{i=1}^{n-1} 2\omega_i\omega_{i+1}.$$

where the c_{ii} are 1 or 2.

For type C_n, the map $\Lambda \to \Lambda/\Lambda_r$ sends ω_i to the class of i. Previous arguments easily handle the n odd case. If n is even, previous arguments leave us to consider the term $2\omega_{n-1}\omega_n$. But

$$\begin{aligned} 2c_1^{K_0}([\mathcal{L}(\omega_{n-1})]) &\equiv 2(c_1^{K_0}(g_{s_{n-1}}) + c_1^{K_0}([\mathcal{L}(\alpha_{n-1})])) \mod \gamma^2(\mathfrak{B}_s) \\ &\equiv c_1^{K_0}(2g_{s_{n-1}}) + 2c_1^{K_0}([\mathcal{L}(\alpha_{n-1})]), \end{aligned}$$

and again we find that $\mathfrak{c}'(\theta)$ is rational.

For type B_n, the map $\Lambda \to \Lambda/\Lambda_r$ sends ω_n to 1 and all other fundamental weights to zero. Consequently, it suffices to consider the term $2\omega_{n-1}\omega_n$ in q. For this we can apply the argument in the preceding paragraph. □

7. Application to essential dimension

We now apply results from the previous section to give a lower bound on the essential dimension $\operatorname{ed}(G)$ for some algebraic groups G. We refer to Reichstein's 2010 ICM lecture [30] for a definition and survey of this notion. Roughly speaking, it gives the number of parameters required to specify a G-torsor.

7.1. **Proposition.** *Let G be an absolutely almost simple algebraic group. Then $\operatorname{ed}(G) \geq 3$ unless G is isomorphic to Sp_{2n} for some $n \geq 2$ (in which case $\operatorname{ed}(G) = 0$) or G has type A.*

The lower bound of 3 is in many cases very weak, but it has the advantage of being uniform and having a proof that is almost as uniform. The existence of the Rost invariant gives the same lower bound on $\operatorname{ed}(G)$ when G is simply connected, so our proposition can be viewed as removing the hypothesis "simply connected" from that result.

7.2. **Remark.** For groups of type A, the lower bound is more complicated and we do not know what the answer is in every case. Of course $\mathrm{ed}(\mathrm{SL}_n) = 0$. Some other known cases are: For n divisible by the square of a prime, we know that $\mathrm{ed}(\mathrm{PGL}_n) \geq 4$ by [29, Th. 16.1(b)]; and for "intermediate" groups of type A in good characteristic, the essential dimension is at least 4 by [31, Th. 8.13]. On the other hand, A.A. Albert conjectured that central simple algebras of prime degree are cyclic, which would imply that $\mathrm{ed}(\mathrm{PGL}_n) = 2$ for square-free n. However, this is only currently known for $n = 2$, 3, and 6.

Proof of Prop. 7.1. *Main case*: Suppose first that G is not of type A, C, nor E_6. As essential dimension only goes down with field extensions, we may assume that k is algebraically closed and bound $\mathrm{ed}(G_s)$ where G_s is a split simple group not of type A or C. Put $\widetilde{G}_s$ for the simply connected cover of G_s. Fix a versal $\widetilde{G}_s$-torsor $\tilde{\xi} \in H^1(L, \widetilde{G}_s)$ for some extension L/k. Let K be a field between k and L of minimal transcendence degree such that there is a $\xi \in H^1(K, G_s)$ whose image in $H^1(L, G_s)$ is the same as the image of $\tilde{\xi}$.

For sake of contradiction, suppose that K has transcendence degree at most 2 over k. By the hypothesis on the type of G, the Tits algebras of ${}_{\xi}G$ have exponent a power of 2 and so are actually of index 1 or 2 over K by De Jong, see [4] or [20, Th. 4.2.2.3]. By Proposition 6.5, there is a class $\psi \in \gamma^2(\mathfrak{B})$ whose image under restriction to L is $\mathfrak{c}'(\theta)$. Now $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}_K)$ is zero by Prop. 3.2 because $H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ is zero, and it follows that $\mathfrak{c}'(\theta)$ is zero in $\operatorname{Tors}\mathrm{CH}^2(\mathfrak{B}_L)$. But $\mathfrak{c}'(\theta)$ has order $N(G_s) \neq 1$, a contradiction.

Type C: If G of type C_n ($n \geq 2$) is simply connected, by hypothesis it is not Sp_{2n}, so the Rost invariant is not zero on a versal G-torsor and the claim follows. So suppose G is adjoint. (We give an argument that is characteristic-free; if $\operatorname{char} k \neq 2$, then $\mathrm{ed}(G) \geq n + 1$ by [3, (1.1)].) If n is odd then we can construct a nonzero normalized cohomological invariant of PSp_{2n} of degree 4 as in [21, Th. 4.1] and that case is settled.

It remains to show that $\mathrm{ed}(\mathrm{PSp}_{2n}) \geq 3$ when k is algebraically closed and n is even. Let k' be an extension of k that has a quaternion division algebra D and fix a class $\zeta \in H^1(k', \mathrm{PSp}_{2n})$ with image $[D] \in H^2(k', \mu_2)$ under the natural connecting homomorphism $\partial_{k'}$. Fix a versal torsor $\tilde{\xi} \in H^1(L, {}_{\zeta}\,\mathrm{Sp}_{2n})$ for some extension L/k'. Let K be a field between k and L so that there is a class $\xi \in H^1(K, \mathrm{PSp}_{2n})$ with the same image as $\tilde{\xi}$ in $H^1(L, \mathrm{PSp}_{2n})$. For sake of contradiction, suppose that K has transcendence degree at most 2 over k.

By De Jong, $\partial_K(\xi)$ is the class of a quaternion algebra in $H^2(K, \mu_2)$. So we could arrange from the beginning that $k' = K$ and $[D] = \partial_K(\xi)$. We have a well-defined invariant

$$\operatorname{im}\left[H^1(*, {}_{\zeta}\,\mathrm{Sp}_{2n}) \to H^1(*, {}_{\zeta}\,\mathrm{PSp}_{2n})\right] \to H^3(*, \mathbb{Z}/2\mathbb{Z})$$

defined on extensions of K because the Rost invariant vanishes on H^1 of the center of every simply connected group of type C_n with n even [12]. On the other hand, the class of ξ belongs to the domain of this map and the invariant does not vanish on it; this contradicts the hypothesis that K has transcendence degree at most 2 over k.

The remaining case of type E_6 is known by arguments as in [13, 9.5, 9.7]. Alternatively, one can repeat the "main case" argument focusing on essential 2-dimension instead of essential dimension. $\square$

Although we stated the proposition for absolutely almost simple groups, it quickly leads to the lower bound $\mathrm{ed}(G) \geq 3$ for more groups G. We mention: (A) to obtain a lower bound on $\mathrm{ed}(G)$ for any linear group G, it suffices to give a lower bound on the essential dimension of the identity component of G [1, 6.19], so one needn't assume that G is connected. (B) If k/k_0 is a finite separable extension, one has $\mathrm{ed}(R_{k/k_0}(G)) \geq \mathrm{ed}(G)$, so the proposition immediately gives a similar (but slightly more complicated to state) result for groups that are simple but not absolutely simple. (C) If $G \simeq G_1 \times G_2$, then $\mathrm{ed}(G) \geq \max\{\mathrm{ed}(G_1), \mathrm{ed}(G_2)\}$, and in this way we can weaken the hypothesis "simple" to "semisimple" at the cost of demanding that G be adjoint or simply connected.

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